Last time : Vector spaces are sets V endowed with

- a notion of addition : $v, w \in V \rightsquigarrow v + w \in V$
- a notion of scalar multiplication : $v \in V, \lambda \in \mathbb{R} \rightsquigarrow \lambda v \in V$
- a notion of zero vector : $0_V \in V$
- a notion of opposites : $v \in V \rightsquigarrow \underset{\substack{\| \\ (-1)v}}{-v} \in V$

Subspaces are subsets $W \subseteq V$ which inherit a vector space structure from that of V

$\hookrightarrow \forall \; w, w' \in W, \; w + w' \in W$

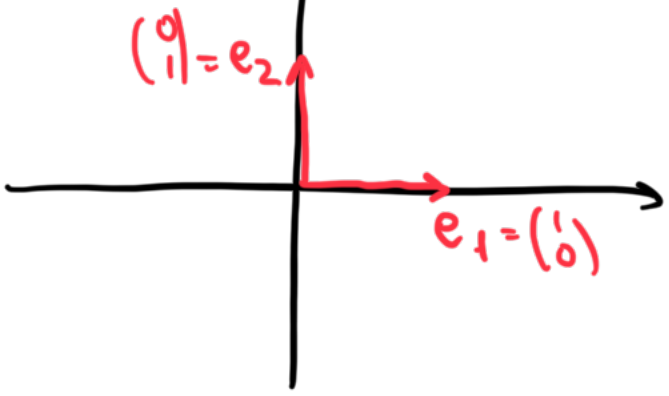
$\hookrightarrow \forall \; w \in W, \lambda \in \mathbb{R}, \; \lambda w \in W$

For any vectors $v_1, \dots, v_k \in V$, the subset span$\{v_1, \dots, v_k\}$ $= \{$ linear combinations of $v_1, \dots, v_k \}$ is a subspace of V.

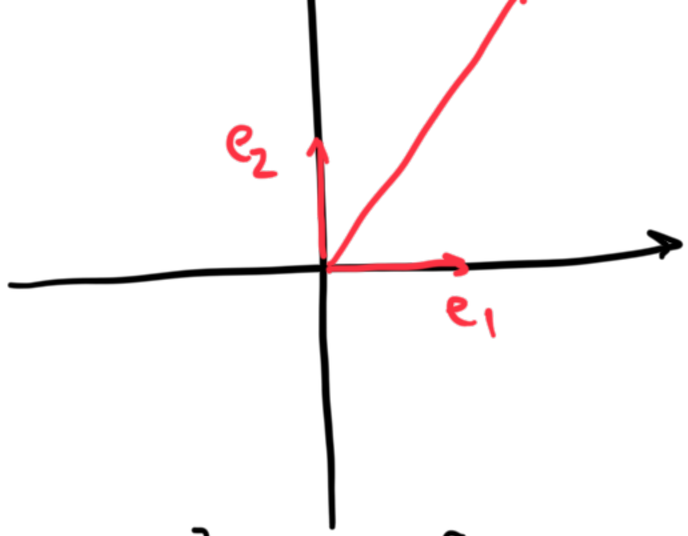
(in this case, we say "the subspace span$\{v_1, \dots, v_k\}$ is spanned/generated by $v_1, \dots, v_k$)

Conversely, any subspace of V is the span of its elements.

Today: $V = \mathbb{R}^2$



$\mathbb{R}^2 = \mathrm{span}\{e_1, e_2\}$
a basis



$\mathbb{R}^2 = \mathrm{span}\{e_1, e_2, v\}$
not a basis

DEF 13.1 : We call vectors $v_1, \ldots, v_k \in V$ a **basis** of V if

- $\mathrm{span}\{v_1, \ldots, v_k\} = V$
- $v_1, \ldots, v_k$ are minimal with this property

follows from Thm on Ex Sheet 1.

$v_1, \ldots, v_k$ are $\underbrace{\text{linearly independent}}$

i.e. $\nexists\ \lambda_1, \ldots, \lambda_k \in \mathbb{R}$ not all 0 s.t. $\lambda_1 v_1 + \ldots + \lambda_k v_k = 0_V$

(by the way, $v_1, \ldots, v_k$ are called linear dependent if
$\exists\ \lambda_1, \ldots, \lambda_k \in \mathbb{R}$ not all 0 s.t. $\lambda_1 v_1 + \ldots + \lambda_k v_k = 0_V$)

New topic: change of basis

- Assume a vector space V has two bases $\begin{matrix} v_1, \ldots, v_n \\ w_1, \ldots, w_m \end{matrix} \in V$

Then $\left\{ w_i = a_{1i} v_1 + \ldots + a_{ni} v_n \right\}_{1 \le i \le m}$ for various constants a_{ji}, b_{ij}

$\left\{ v_j = b_{1j} w_1 + \ldots + b_{mj} w_m \right\}_{1 \le j \le n}$

Define the Change of basis matrices

$A = (a_{ji})_{1\le i\le m,\, 1\le j\le n} \in \mathbb{R}^{n\times m}$ from w basis to v basis

$B = (b_{ij})_{1\le i\le m,\, 1\le j\le n} \in \mathbb{R}^{m\times n}$ from v basis to w basis

Let's figure out the relation between these matrices

$$\forall j:\ v_j = b_{1j} w_1 + \dots + b_{mj} w_m = \begin{array}{c} b_{1j}(a_{11}v_1 + \dots + a_{n1}v_n) \\ + \\ \vdots \\ + \\ b_{mj}(a_{1m}v_1 + \dots + a_{nm}v_n) \end{array} =$$

$$= v_1(\underbrace{a_{11}b_{1j} + a_{12}b_{2j} + \dots + a_{1m}b_{mj}}_{(AB)_{1j}}) + \dots + v_n(\underbrace{a_{n1}b_{1j} + \dots + a_{nm}b_{mj}}_{(AB)_{nj}})$$

Notation: $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ & & & \end{pmatrix}$, $B = \begin{pmatrix} & b_{1j} & \\ & \vdots & \\ & b_{mj} & \end{pmatrix}$, $(AB)_{1j} = a_{11}b_{1j} + \dots + a_{1m}b_{mj}$

Lemma: if $v_1, \dots, v_k \in V$ are linearly independent, then

$\lambda_1 v_1 + \dots + \lambda_k v_k = \mu_1 v_1 + \dots + \mu_k v_k \Rightarrow \lambda_1 = \mu_1, \dots, \lambda_k = \mu_k$

$(AB)_{1j} = 0 \dots\dots (AB)_{jj} = 1 \dots\dots (AB)_{nj} = 0 \quad \forall\, 1\le j\le n.$

$$\Downarrow$$

$$AB = I_n$$

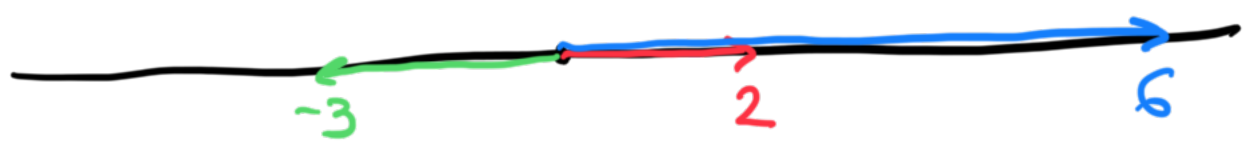
Similarly, one obtains $BA = I_m$ $\Bigg\} \Rightarrow$ $m = n$ (i.e. square) and $B = A^{-1}$

THM 13.2: The change of basis matrices A and B from one basis to another basis and back are **inverses**

COR 13.3: any two bases of a vector space have the same number of vectors

DEF 13.4: the **dimension** of a vector space V is the number of vectors in <u>any</u> basis of V.

Example: $V = \mathbb{R}^1$



$$\longleftarrow \; = -\frac{3}{2} \cdot \longrightarrow$$

$$\rightarrow \; = \frac{1}{3} \cdot \longrightarrow$$

while these three vectors span $\mathbb{R}^1$, they do not form a basis

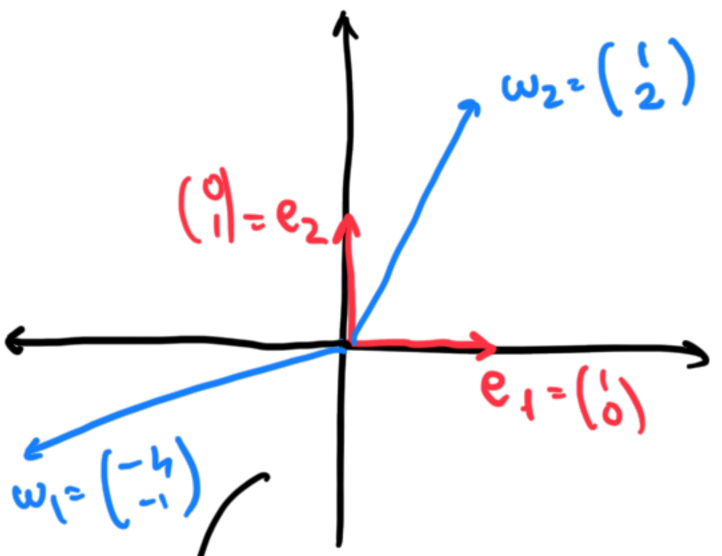
on the other hand, $\{\leftarrow\}$ forms a basis of $\mathbb{R}^1$

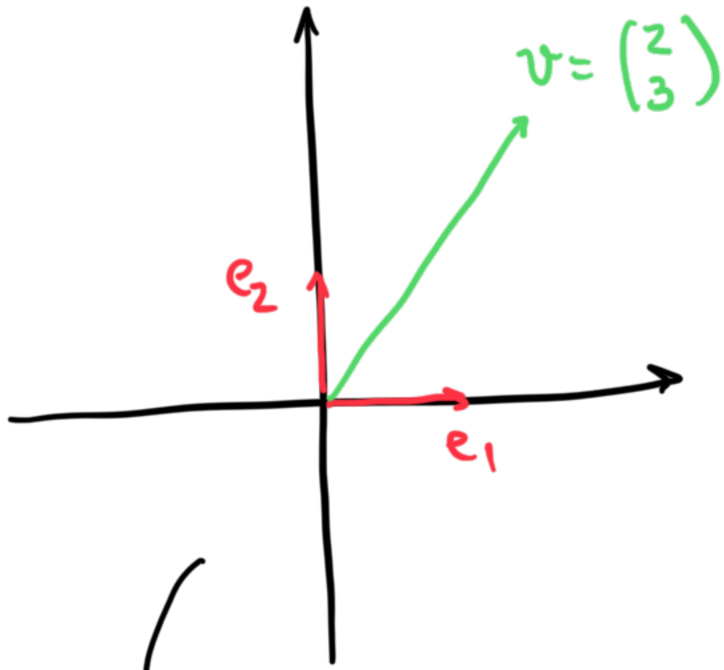
$\{\rightarrow\}$ forms a basis of $\mathbb{R}^1$

$\{\rightarrow\}$ forms a basis of $\mathbb{R}^1$

$\dim(\mathbb{R}^1) = 1$

Example: $V=\mathbb{R}^2$





$\{e_1, e_2\}$ form a basis

$\Downarrow$

$\dim(\mathbb{R}^2)=2$

not a basis: $v = 2e_1+3e_2$

$\{w_1, w_2\}$ form another basis

$$w_1 = -4\,e_1 - 1\,e_2$$
$$w_2 = 1\,e_1 + 2\,e_2$$

Change of basis matrix from w to e is

$$A=\begin{pmatrix}-4 & 1\\ -1 & 2\end{pmatrix}$$

Change of basis matrix from e to w is

$$A^{-1}=\frac{1}{7}\begin{pmatrix}-2 & 1\\ -1 & 4\end{pmatrix}$$

$$e_1 = -\frac{2}{7}\,w_1 - \frac{1}{7}\,w_2$$
$$e_2 = \frac{1}{7}\,w_1 + \frac{4}{7}\,w_2$$

In general, $\dim(\mathbb{R}^n)=n$

standard basis $e_1=\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}, e_2=\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \ldots\ldots\; e_n=\begin{pmatrix}0\\0\\0\\\vdots\\1\end{pmatrix}$

- Question: if $w_1, \dots, w_m \in \mathbb{R}^n$, when do they form a basis?
  - need $m = n$, but not always enough.
  - $w_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} = a_{11}e_1 + a_{21}e_2 + \dots + a_{n1}e_n$

    $\vdots$

    $w_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} = a_{1n}e_1 + a_{2n}e_2 + \dots + a_{nn}e_n$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$
$\qquad\ \ \underbrace{\ \ }_{w_1} \qquad \underbrace{\ \ }_{w_n}$

is the change of basis matrix from w-basis to the e-basis

$w_1, \dots, w_n$ form a basis of $\mathbb{R}^n \iff A$ is invertible $\iff \det(A) \neq 0$

Ex: do $\begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 9 \\ -16 \end{pmatrix}$ form a basis of $\mathbb{R}^2$?

$A = \begin{pmatrix} 7 & 9 \\ 5 & -16 \end{pmatrix}$, $\det(A) = -157 \neq 0 \implies$ basis ✓

New topic: coordinate systems (of $\mathbb{R}^n$ or in any vector space V)

Start with a basis $v_1, \dots, v_n$ of any vector space V.

$\Downarrow$

any $w \in V$ can be written uniquely as

$x_1 v_1 + \dots + x_n v_n$ where $x_1, \dots, x_n \in \mathbb{R}$

$w = x_1 v_1 + \dots + x_n v_n$, where

$x_1, \dots, x_n$ are called the coordinates of w relative to the basis $v_1, \dots, v_n$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$ is also the "coordinate vector" of w rel. to basis $v_1, \dots, v_n$

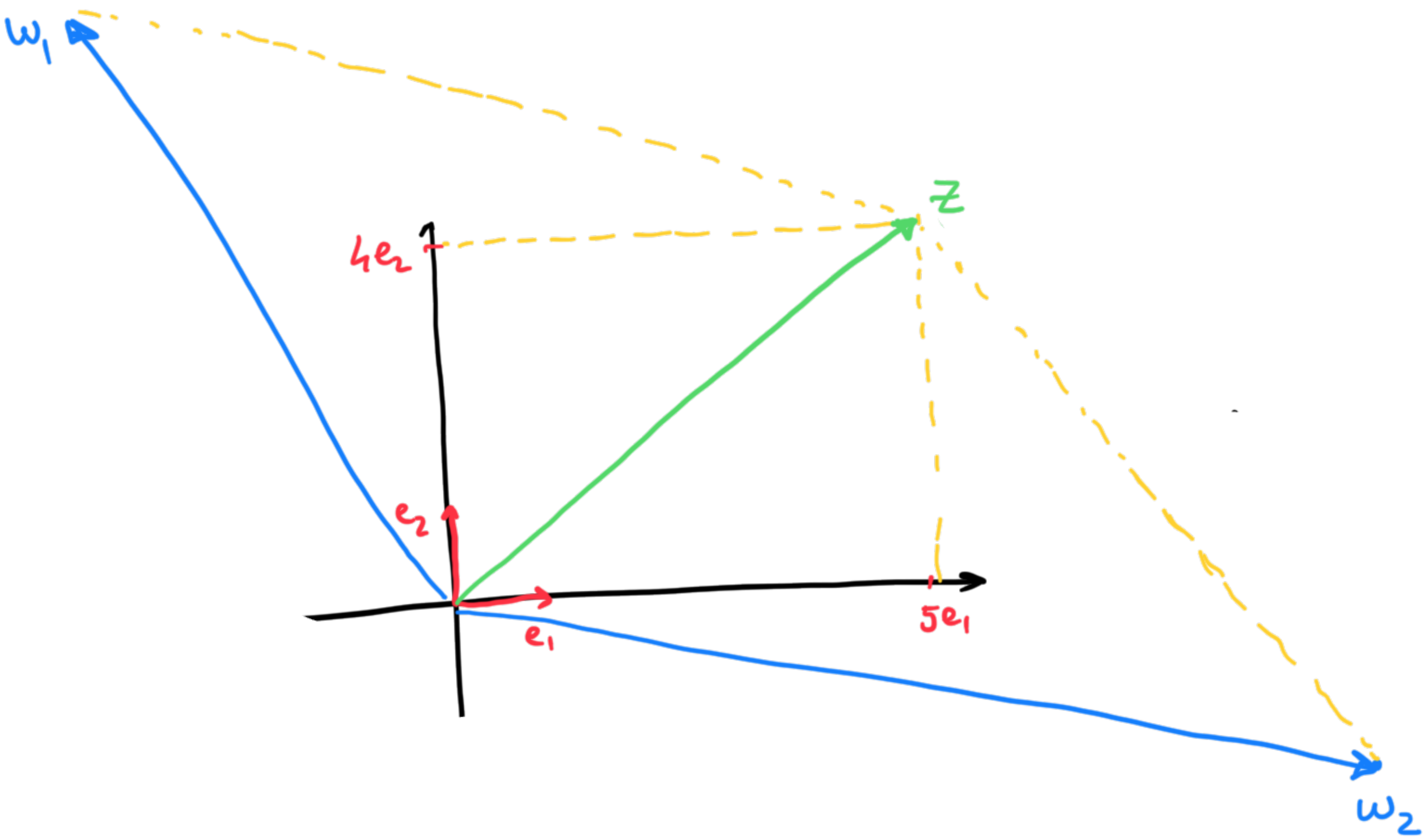
Example: $(V = \mathbb{R}^2)$

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ Z = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 5\, v_1 + 4\, v_2$$

coordinates rel v basis

$$w_1 = \begin{pmatrix} -7 \\ 6 \end{pmatrix},\ w_2 = \begin{pmatrix} 12 \\ -2 \end{pmatrix},\ Z = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 1\, w_1 + 1\, w_2$$

coordinates rel w basis



How to change coordinates?

$\{v_1, \dots, v_n\}$ V $x_1 v_1 + \dots + x_n v_n$

bases of $V \ni Z$

$\{w_1, \dots, w_n\}$ $\quad Z = y_1 w_1 + \dots + y_n w_n$

Use Change of basis matrix: $w_j = a_{1j} v_1 + \dots + a_{nj} v_n$

$A = (a_{ij})_{1 \le i,j \le n}$

$$Z = y_1 w_1 + \dots + y_n w_n = y_1(a_{11} v_1 + \dots + a_{n1} v_n) + \dots + y_n (a_{1n} v_1 + \dots + a_{nn} v_n)$$

$$= v_1 \underbrace{(a_{11} y_1 + a_{12} y_2 + \dots + a_{1n} y_n)}_{x_1} + \dots + v_n \underbrace{(a_{n1} y_1 + \dots + a_{nn} v_n)}_{x_n}$$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} y_1 + \dots + a_{1n} y_n \\ \vdots \\ a_{n1} y_1 + \dots + a_{nn} y_n \end{pmatrix} = A \cdot Y = A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$\Downarrow$

THM 13.5 (coordinate change formula)

$$X = A Y$$

coordinates w.r.t v basis

change of basis matrix from w basis to v basis

coordinates w.r.t. w basis

Example: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $z = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 5 v_1 + 4 v_2$

$(V = \mathbb{R}^2)$

$$w_1 = \begin{pmatrix} -7 \\ 6 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 12 \\ -2 \end{pmatrix}, \quad z = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 1\, w_1 + 1\, w_2$$

$A = \begin{pmatrix} -7 & 12 \\ 6 & -2 \end{pmatrix}$, $X = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$, $Y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ checks out $X = AY$

New topic: Changing between multiple bases

$v_1, \dots, v_n$

$w_1, \dots, w_n$ bases of V

$z_1, \dots, z_n$

$w_j = a_{1j} v_1 + \dots + a_{nj} v_n \rightsquigarrow A = (a_{ij})$ changes from basis w to v

$z_k = b_{1k} w_1 + \dots + b_{nk} w_n \rightsquigarrow B = (b_{ij})$ changes from basis z to w

THM 13.6: the matrix $C = AB$ changes from the basis z directly to the basis v

Proof:

$$z_k = b_{1k} w_1 + \dots + b_{nk} w_n = \begin{array}{c} b_{1k}(a_{11} v_1 + \dots + a_{n1} v_n) \\ + \\ \vdots \\ + \\ b_{nk}(a_{1n} v_1 + \dots + a_{nn} v_n) \end{array}$$

$$= v_1\underbrace{(a_{11}b_{1k}+\dots+a_{1n}b_{nk})}_{(AB)_{1k}}+\dots+v_n\underbrace{(a_{n1}b_{1k}+\dots+a_{nn}b_{nk})}_{(AB)_{nk}}$$

Let $C = AB = (c_{ij})_{1\leq i,j\leq n}$; therefore, we have

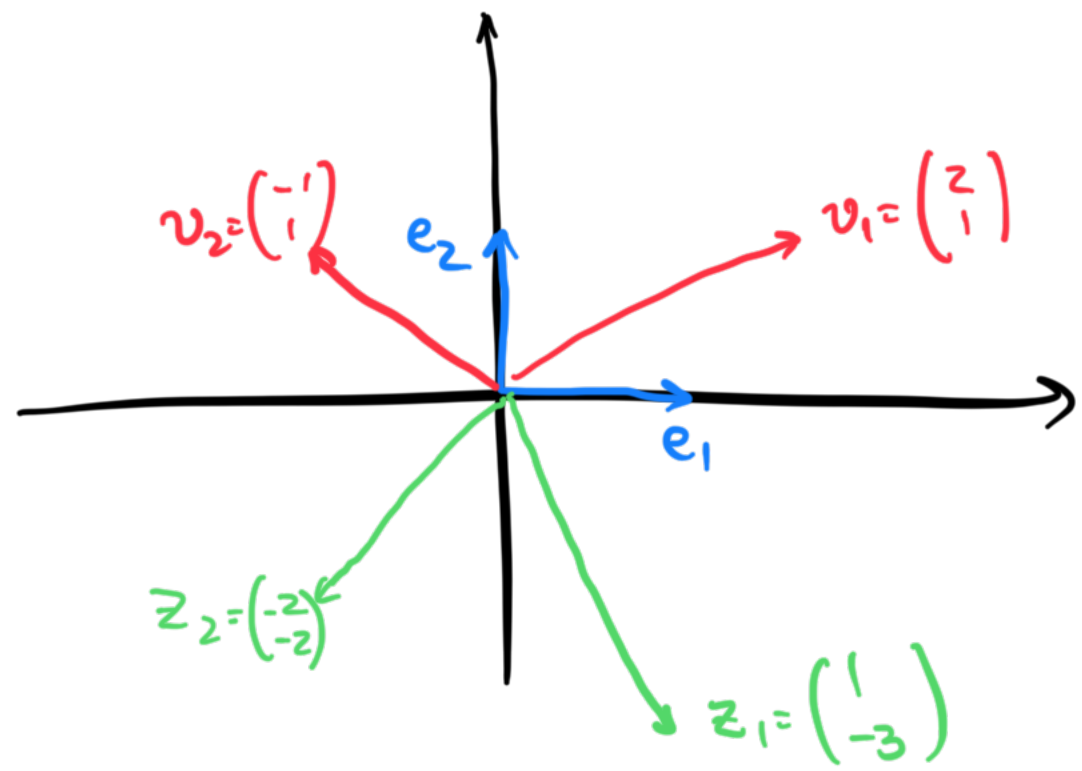
$$z_k = c_{1k}v_1 + \dots + c_{nk}v_k \qquad \square$$

Schematically :

$$\begin{tikzcd}
v_1,\dots,v_n \arrow[d, bend left, "A^{-1}"] \arrow[dd, bend left=80, "B^{-1}A^{-1}=(AB)^{-1}"] \\
w_1,\dots,w_n \arrow[u, bend left, "A"] \arrow[d, bend left, "B^{-1}"] \\
z_1,\dots,z_n \arrow[u, bend left, "B"] \arrow[uu, bend left=80, "AB"]
\end{tikzcd}$$

Application : how to change basis between arbitrary bases of $\mathbb{R}^n$

(say $n=2$)



$A=\begin{pmatrix}2 & -1\\ 1 & 1\end{pmatrix}$ changes from red basis to blue basis

$\begin{pmatrix}1 & -2\end{pmatrix}$ changes from green basis to blue basis

$B = \begin{pmatrix} 1 & -2 \\ -3 & -2 \end{pmatrix}$ changes from green basis to blue

$\Downarrow$

$A^{-1}B = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & -\frac{4}{3} \\ -\frac{7}{3} & -\frac{2}{3} \end{pmatrix}$ changes from green to red

i.e. $z_1 = c_{11} v_1 + c_{21} v_2 = -\frac{2}{3} v_1 - \frac{7}{3} v_2$

$z_2 = c_{12} v_1 + c_{22} v_2 = -\frac{4}{3} v_1 - \frac{2}{3} v_2$

$\Downarrow$

Sanity check: $\begin{pmatrix} 1 \\ -3 \end{pmatrix} = -\frac{2}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{7}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ✓

$\begin{pmatrix} -2 \\ -2 \end{pmatrix} = -\frac{4}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ✓